# Quadratic variations and parameter estimation for stochastic heat equation with additive fractional noise

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Center for Applied Mathematics, Tianjin University

# Outline

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§1 Background

§2 The fractional (weighted) quadratic variations

§3 Parameter estimation based on temporal quadratic variation
(Time sampling at a fixed space point)

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- In recent ten years, there have been many papers on the parameter estimation of stochastic heat equations.

[1] M. Bibinger and M. Trabs, On central limit theorems for power variations of the solution to the stochastic heat equation, *Stochastic models*, Springer Proc. Math. Stat. 294, 69-84 (2019).

[2] C. Chong, High-frequency analysis of parabolic stochastic PDEs, *Ann. Statist.* **48** (2020), 1143-1167.

[3] C. Chong and R-C. Dalang, Power variations in fractional sobolev spaces for a class of parabolic stochastic PDEs, *Bernoulli*, **29** (2023), 1792-1820.

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- Some results:

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- J. Pospisil and R. Tribe (SAA, 2007) considered parameter estimation and exact variations of the equation

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \theta\sigma(u)\dot{W}(t,x), \quad t \geq 0, \quad x \in \mathbb{R}$$

with $u(0,x) = \varphi(x)$, where $\dot{W}$ is a white noise, $\sigma$ is a Lipschitz function and $\theta > 0$ is a parameter. They showed

$$\mathbf{V}^4(u^x;[s,t]) := \lim_{n\to\infty}\sum_{j=1}^{n}(u(t_j,x)-u(t_{j-1},x))^4 = \frac{3\theta}{\pi}\int_s^t \sigma(u(r,x))^4 dr$$

with $0 \leq s < t$ and $t_j - t_{j-1} = \frac{t-s}{n}$, and

$$\mathbf{V}^2(u^t;[a,b]) := \lim_{n\to\infty}\sum_{j=1}^{n}(u(t,x_j)-u(t,x_{j-1}))^2 = \frac{\theta}{2}\int_a^b \sigma(u(t,y))^2 dy$$

with $a < b$ and $x_j - x_{j-1} = \frac{b-a}{n}$ in probability. As applications, they introduced the estimators of $\theta$ and showed the weak consistency. However, they did not establish the asymptotic normality.

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[5] J. Pospisil and R. Tribe, Parameter estimation and exact variations for stochastic heat equations driven by space-time white noise, *Stoch. Anal. Appl.* **4** (2007), 830-856.

- I. Cialenco and Y. Huang (SD, 2020) considered parameter estimation on the SPDE

$$\frac{\partial}{\partial t}u(t,x) = \theta\Delta u(t,x)dt + \sigma\dot{W}(t,x), \quad t \geq 0, \quad x \in \mathbb{R}$$

with $u(0,x) = 0$, where $\sigma, \theta > 0$ are two parameters. On a finite sampling interval, they introduced the estimators of $\theta$ and $\sigma^2$ and their the asymptotic behavior of the estimators.

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- Time sampling at a fixed space point $x$:

$$\hat{\theta}_{n,x} := \frac{3(d-c)\sigma^4}{\pi\sum_{j=1}^{n}\left(u(t_j,x) - u(t_{j-1},x)\right)^4}$$

  and

$$\widehat{\sigma^2}_{n,x} := \sqrt{\frac{\theta\pi}{3(d-c)}\sum_{j=1}^{n}\left(u(t_j,x) - u(t_{j-1},x)\right)^4},$$

  where $t_j = c + \frac{j}{n}(d-c), j = 0,1,2,\ldots,n$ with $[c,d] \subset [0,\infty)$.

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For the above estimators, they obtained the following asymptotic normalities:

$$\sqrt{n}\left(\hat{\theta}_{n,x}-\theta\right)\longrightarrow N\left(0,\frac{1}{9}\theta^2 C_2\right)$$

and

$$\sqrt{n}\left(\widehat{\sigma^2}_{n,x}-\sigma^2\right)\longrightarrow N\left(0,\frac{1}{36}\sigma^4 C_2\right)$$

in distribution, as $n$ tends to infinity.

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- Space sampling at a fixed time instance $t$:

$$\overline{\theta}_{m,t}:=\frac{(b-a)\sigma^2}{2\sum_{j=1}^m\left(u(t,x_j)-u(t,x_{j-1})\right)^2}$$

and

$$\overline{\sigma^2}_{m,t}:=\sqrt{\frac{2\theta}{b-a}\sum_{j=1}^m\left(u(t,x_j)-u(t,x_{j-1})\right)^2},$$

where $x_j=a+\frac{j}{m}(b-a), j=0,1,2,\dots,m$ with $[a,b]\subset\mathbb{R}$.

- Space-time sampling and joint estimation of $\theta$ and $\sigma$:

$$\tilde{\theta}_{n,m} := \frac{\pi(b-a)^2 V_n^4(u^x;[c,d])}{12(d-c)^2(V_m^2(u^t;[a,b]))^2} \longrightarrow \theta$$

and

$$\widetilde{\sigma^2}_{n,m} := \frac{\pi(b-a) V_n^4(u^x;[c,d])}{6(d-c) V_m^2(u^t;[a,b])} \longrightarrow \sigma^2$$

in probability, as $n, m \to \infty$.

- I. Cialenco and H. Kim (SPA. 2022), authors considered the equation:

$$\frac{\partial u}{\partial t} = \theta \Delta u(t,x) + \sigma \dot{W}(x), \quad t \geq 0, \quad , \quad x \in G$$

with $u(0,x) = 0$. They introduced the estimators

$$\widetilde{\theta^2}_n := \frac{\sigma^2 (b-a)}{\sum_{i=1}^{n} (u_x(t,x_i) - u_x(t,x_{i-1}))^2}$$

and

$$\widetilde{\sigma^2}_n := \frac{\theta^2 \sum_{i=1}^{n} (u_x(t,x_i) - u_x(t,x_{i-1}))^2}{b-a}$$

[7] I. Cialenco and H. Kim, Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise, *Stochastic Processes Appl.* **143** (2022), 1-30.

F. Hildebrandt and M. Trabs, Parameter estimation for SPDEs based on discrete observations in time and space, *Electron. J. Stat.* **15** (2021), 2716-2776.

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I. Cialenco and L. Xu, A note on error estimation for hypothesis testing problems for some linear SPDEs, *Stoch PDE: Anal. Comp.* **2** (2014), 408-431.

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[8] H. Ouahhabi and Ciprian A. Tudor (2013 , Additive Functionals of the Solution to Fractional Stochastic Heat Equation, *J. Fourier Anal. Appl.* **19** (2013), 777-791.

- These studies basically sample and construct estimators within a finite interval, and basically, estimators are constructed using time sampling and spatial sampling separately.
- On the other hand, Ouahhabi and Tudor (JFAA, 2013) considered the equation

$$\frac{\partial}{\partial t}u^H(t,x)=\frac{\partial^2}{\partial x^2}u^H(t,x)+\dot{W}^H(t,x),\quad t\geq 0, x\in\mathbb{R},$$

with $\frac{1}{2}<H<1$, where $W^H=\{W^H(t,x),t\geq 0,x\in\mathbb{R}\}$ is the fractional noise. They showed that

$$E\left(u^H(t,x)u^H(s,x)\right)=\frac{H(2H-1)}{\sqrt{2\pi}}\int_0^t\int_0^s|u-v|^{2H-2}\frac{dvdu}{\sqrt{(t+s)-(u+v)}},$$

$$c_H|t-s|^{2H-\frac{1}{2}}\leq E\left[\left(u^H(t,x)-u^H(s,x)\right)^2\right]\leq C_H|t-s|^{2H-\frac{1}{2}},$$

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- They also showed that the temporal process $\{u^H(t,\cdot), t\geq 0\}$ is $\rho$-local nondeterministic and introduced existence and regularity of local time.

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- S. Torres, C-A. Tudor, F-G. Viens (EJP, 2014) considered the above equation with $W^H = \{W^H(t,x), t \geq 0, x \in \mathbb{R}\}$ being a fractional-colored Gaussian noise with Hurst index $H \in (\frac{1}{2}, 1)$ in the time variable and spatial covariance function $f$ which is the Fourier transform of a tempered measure $\mu$.

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- They introduced the central and non-central limit theorems associated with quadratic variations:

$$V_n^H(t,x) = \sum_{j=1}^{n} \left\{ \frac{\left(u^H(t_j,x) - u^H(t_{j-1},x)\right)^2}{E\left(u^H(t_j,x) - u^H(t_{j-1},x)\right)^2} - 1 \right\}.$$

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As application, they introduced the estimator of $H$.

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- Prompted by these results, in this talk we also consider the equation

$$\frac{\partial}{\partial t}u^H(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u^H(t,x) + \sqrt{\theta}\dot{W}^H(t,x), \quad t \geq 0, x \in \mathbb{R},$$

with $u^H(0,x) = 0$ and $\frac{1}{2} < H < 1$, where $\theta > 0$ is a parameter and $W^H = \{W^H(t,x), t \geq 0, x \in \mathbb{R}\}$ is the fractional noise.

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Our purpose is as follows:

- to establish a fractional (weighted) quadratic variation

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Our purpose is as follows:

- to establish a fractional (weighted) quadratic variation
- to establish the estimator of $\theta$ basad on the quadratic variation

- Clearly, we have

$$u^H(t,x) = \sqrt{\theta}\int_0^t \int_{\mathbb{R}} G(t-r, x-y) W^H(dr, dy)$$

with $x \in \mathbb{R}$ and $t \geq 0$, where $G(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ is the heat kernel.

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- Denote the temporal process by $u^x=\{u^H(t,x),t\geq 0\}$. By H. Ouahhabi and Ciprian A. Tudor (JFAA, 2013) we then have

$$[u^x,u^x]_t=\begin{cases}0, & H>\frac{3}{4};\\ C\theta t, & H=\frac{3}{4};\\ +\infty, & \frac{1}{2}<H<\frac{3}{4}\end{cases}$$

for all $t>0$.

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for all $t > 0$.

- Bi-fractional Brownian motion $B^{H,K} = \{B_t^{H,K}, t \geq 0\}$ with $0 < H < 1$, $0 < K < 2$ and $0 < HK < 1$: a central Gaussian process with

$$E\left[B_t^{H,K} B_s^{H,K}\right] = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right)$$

Our starting point is the following definition.

**Definition**

Denote $\kappa = H - \frac{1}{4}$. Assume that integral

$$I_\varepsilon^H(f,t,x) = \frac{1}{\varepsilon^{2\kappa}} \int_0^t \{f\left(u^H(s+\varepsilon,x)\right) - f(u^H(s,x))\}\{u^H(s+\varepsilon,x) - u^H(s,x)\}ds^{2\kappa}$$

exists for all $\varepsilon > 0$, $t \geq 0$ and $x \in \mathbb{R}$, where $f$ is a Borel measurable function on $\mathbb{R}$. The limit

$$[f(u^x), u^x]_t^{(TQ)} := \lim_{\varepsilon \to 0} I_\varepsilon(f,t,x)$$

is called the fractional quadratic covariation of of $f(u^x)$ and $u^x$, provided the limit exists in probability.

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Denote $\kappa = H - \frac{1}{4}$. Assume that integral

$$I_\varepsilon^H(f,t,x) = \frac{1}{\varepsilon^{2\kappa}} \int_0^t \{f\left(u^H(s+\varepsilon,x)\right) - f(u^H(s,x))\}\{u^H(s+\varepsilon,x) - u^H(s,x)\} ds^{2\kappa}$$

exists for all $\varepsilon > 0$, $t \geq 0$ and $x \in \mathbb{R}$, where $f$ is a Borel measurable function on $\mathbb{R}$. The limit

$$[f(u^x), u^x]_t^{(TQ)} := \lim_{\varepsilon \to 0} I_\varepsilon(f,t,x)$$

is called the fractional quadratic covariation of of $f(u^x)$ and $u^x$, provided the limit exists in probability.

- For a continuous adapted process $X = \{X_t, t \geq 0\}$, the quadratic covariation $[f(X), X]$ of the process $X$ and $f(X)$ is defined as follows:

$$[f(X), X] := \frac{1}{\varepsilon} \int_0^t \{f(X_{s+\varepsilon}) - f(X_s)\}\{X_{s+\varepsilon} - X_s\} ds$$

Remark: It also is important to note that the temporal quadratic covariation

$$[f(u^x), u^x]^{(TQ)}$$

can be defined as the limit in probability

$$\lim_{n\to\infty} n^{2\kappa-1}\sum_{j=1}^{n}\{f\left(u^H(t_j,x)\right)-f\left(u^H(t_{j-1},x)\right)\}\{u^H(t_j,x)-u^H(t_{j-1},x)\}$$

for all $t>0$ and $x\in\mathbb{R}$, where $t_j=\frac{j}{n}t$.

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for all $t>0$ and $x\in\mathbb{R}$, where $t_j=\frac{j}{n}t$.

- In general, for a continuous adapted process $X=\{X_t,t\geq 0\}$ the quadratic covariation $[f(X),X]$ of the process $X$ and $f(X)$ is defined as follows:

$$[f(X),X]_t=\lim_{n\to\infty}\sum_{j=1}^{n}\{f(X_{t_j})-f(X_{t_{j-1}})\}\{X_{t_j}-X_{t_{j-1}}\}$$

provided this limit exists in probability.

Remark: It also is important to note that the temporal quadratic covariation

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for all $t > 0$ and $x \in \mathbb{R}$, where $t_j = \frac{j}{n}t$.

- In general, for a continuous adapted process $X = \{X_t, t \geq 0\}$ the quadratic covariation $[f(X), X]$ of the process $X$ and $f(X)$ is defined as follows:

$$[f(X), X]_t = \lim_{n\to\infty} \sum_{j=1}^{n} \{f(X_{t_j}) - f(X_{t_{j-1}})\}\{X_{t_j} - X_{t_{j-1}}\}$$

provided this limit exists in probability.

- The QC $[f(u^x), u^x]^{(TQ)}$ should be called the weighted quadratic covariation which is a simple extension of the classical quadratic covariation.

**Proposition (1)**

*Let* $\frac{1}{2} < H < 1$ *and let* $f \in C^1(\mathbb{R})$. *Then, we have*

$$[f(u^x), u^x]_t^{(TQ)} = \theta K_H \int_0^t f'(u^H(x,s))ds^{2\kappa},$$

*for all* $t \geq 0$ *and in particular we have*

$$[u^x, u^x]_t^{(TQ)} = \theta K_H t^{2\kappa}$$

*for all* $t \geq 0$, *where*

$$K_H = \frac{H}{2\sqrt{2\pi}} \left( 2^{2\kappa}\mathbb{B}(2H, \frac{1}{2}) - \frac{1}{2\kappa}\left(2^{2\kappa} - 1\right) \right)$$

- Take $\theta = 1$ and consider the set

$$\mathscr{H} = \{f : \text{ measurable functions on } \mathbb{R} \text{ such that } \|f\|_{\mathscr{H}} < \infty\},$$

where

$$\|f\|_{\mathscr{H}} := \sqrt{\int_0^T \int_{\mathbb{R}} |f(x)|^2 e^{-\frac{x^2}{2K_H s^{2\kappa}}} \frac{\sqrt{K_H}}{\sqrt{2\pi} s^{1-\kappa}} dx ds}.$$

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- Then, $\mathscr{H}$ is a Banach space with the norm $\|\cdot\|_{\mathscr{H}}$ and the set $\mathscr{E}$ of elementary functions of the form

$$f_{\triangle}(x) = \sum_i f_i 1_{(x_{i-1}, x_i]}(x)$$

is dense in $\mathscr{H}$, where $\{x_i, 0 \le i \le l\}$ is an finite sequence of real numbers such that $x_i < x_{i+1}$.

**Theorem (2)**

*Let* $\frac{1}{2} < H < 1$ *and* $f \in \mathscr{H}$*. Then the quadratic covariation* $[f(u^x), u^x]^{(TQ)}$ *exists and*

$$E\left|[f(u^x), u^x]_t^{(TQ)}\right|^2 \leq C_H \|f\|_{\mathscr{H}}^2 \qquad (0.1)$$

*for all* $t \geq 0$*.*

- On the other hand, Alós *et al.* (2001,AOP) introduced the following Itô formula:

$$F(G_t) = F(0) + \int_0^t F'(G_s)dG_s + \frac{1}{2}\int_0^t F''(G_s)d\varphi(s)$$

for all $t \in [0, T]$ and, where $G = \{G_t, t \geq 0\}$ is a Gaussian process with some suitable conditions, $\varphi(s) = EG_s^2$ is increasing and $F \in C^2(\mathbb{R})$ satisfying

$$|F(x)|, |F'(x)|, |F''(x)| \leq Ce^{Kx^2} \quad (x \in \mathbb{R})$$

with $K \leq \frac{1}{4}\left(\sup_{0\leq t\leq T} \varphi(t)\right)^{-1}$.

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**Theorem (3)**

*Let $\frac{1}{2} \leq H < 1$ and let $F$ be an absolutely continuous function such that the derivative $F' \in \mathscr{H}$ is left continuous. Then we have*

$$F(u^H(t,x) = F(0) + \int_0^t F'(u^H(s,x))u(ds,x) + \frac{1}{2}\left[F'(u^x), u^x\right]_t^{(TQ)}$$

*for all $t \geq 0$ and $x \in \mathbb{R}$.*

[10] E. Alós, O. Mazet and D. Nualart, Stochastic calculus with respect to Gaussian processes, *Ann. Probab.* **29** (2001), 766-801.

Moreover, by using the result obtained Ouahhabi and Tudor (JFAA, 2013), we have known that the weighted local time

$$\mathscr{L}^x(t,y) = 2\alpha K_\alpha \int_0^t \delta(u^H(s,x) - y) s^{2\alpha-1} ds$$

exists in $L^2$ and it is continuous in $(t,y)$.

**Lemma**

*Given $x \in \mathbb{R}$. Then, the integral*

$$\int_{\mathbb{R}} f_\triangle(y) \mathscr{L}^x(dy,t) := \sum_j f_j \left[\mathscr{L}^x(a_j,t) - \mathscr{L}^x(a_{j-1},t)\right]$$

*exists for any $f_\triangle = \sum_j f_j 1_{(a_{j-1},a_j]} \in \mathscr{E}$, and*

$$\int_{\mathbb{R}} f_\triangle(y) \mathscr{L}^x(dy,t) = -\left[f_\triangle(u^x), u^x\right]_t^{(TQ)} \tag{0.2}$$

*for all $t \geq 0$ and $x \in \mathbb{R}$, where $\mathscr{L}^x$ denotes the weighted local time of $u^x$.*

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[8] H. Ouahhabi and Ciprian A. Tudor, Additive Functionals of the Solution to Fractional Stochastic Heat Equation, *J. Fourier Anal. Appl.* **19** (2013), 777-791.

Thanks to the denseness of $\mathscr{E}$ in $\mathscr{H}$, we can then extend the definition of integration with respect to $y \mapsto \mathscr{L}^x(y,t)$ to the elements of $\mathscr{H}$ in the following manner:

$$\int_{\mathbb{R}} f(y)\mathscr{L}^x(dy,t) := \lim_{n\to\infty} \int_{\mathbb{R}} f_{\triangle,n}(y)\mathscr{L}^x(dy,t)$$

in $L^2$ for $f \in \mathscr{H}$ provided $f_{\triangle,n} \to f$ in $\mathscr{H}$, as $n$ tends to infinity, where $\{f_{\triangle,n}\} \subset \mathscr{E}$.

The limit does not depend on the choice of the sequences $\{f_{\triangle,n}\}$ and it represents the integral of $f$ with respect to $\mathscr{L}^x$.

**Theorem (4)**

*Let $f \in \mathscr{H}$. Then the integral*

$$\int_{\mathbb{R}} f(y)\mathscr{L}^x(dy,t)$$

*is well-defined and the Bouleau-Yor type identity*

$$[f(u^x),u^x]_t^{(TQ)} = -\int_{\mathbb{R}} f(y)\mathscr{L}^x(dy,t)$$

*holds for all $t \geq 0$ and $x \in \mathbb{R}$.*

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$$[f(u^x), u^x]_t^{(TQ)} = -\int_{\mathbb{R}} f(y)\mathscr{L}^x(dy,t)$$

*holds for all $t \geq 0$ and $x \in \mathbb{R}$.*

- The above results are also true for more general Gaussian processes $G = \{G_t, t \geq 0\}$ such that $G_0 = 0$, $E[G_t] = 0$,

$$t \mapsto E\left[G_t^2\right] = \varphi(t)$$

is increasing, Hölder continuous and

$$E\left[(G_t - G_s)^2\right] \asymp \varphi(t-s)$$

for all $t > s \geq 0$.

- Some earlier studies for integration with respect to local time:

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C. Feng and H. Zhao, Local time rough path for Lévy processes, *Elect. J. Probab.* **15** (2010), 452-483.

✠ Consider the equation

$$\frac{\partial}{\partial t}u^H(t,x) = \frac{1}{2}\Delta u^H(t,x) + \sqrt{\theta}\dot{W}^H(t,x), \quad t \geq 0, \quad x \in \mathbb{R}$$

with $u(0,x) = 0$, where $\theta > 0$ is a unknown parameter and $W^H$ is the fractional noise with Hurst index $H \in (\frac{1}{2}, 1)$.

✠ Let the temporal process be observed at some discrete time instants $\{t_j = jh, j = 0, 1, 2, \ldots, n\}$ with $h = h(n,t) \to 0$ as $n$ tends to infinity. For all $t > 0$ and $x \in \mathbb{R}$ we denote

$$I_n^H(t,x) := \sum_{j=1}^{n} \left\{ u^H(t_j,x) - u^H(t_{j-1},x) \right\}^2.$$

✠ Two Cases:

(I) $t_n = nh = t$;

(II) $t_n = nh \to \infty$.

✠ Case 1: $h = \frac{t}{n}$. Then, we have

$$n^{2\kappa-1} I_n^H(t,x) \xrightarrow{P} \theta K_H t^{2\alpha}$$

for all $t > 0$ and $x \in \mathbb{R}$. In fact, we can also show that the convergence is almost sure. Thus, we get a strongly consistent estimator of $\theta$ as follows

$$\hat{\theta}_n = \frac{n^{2H-\frac{3}{2}}}{K_H t^{2H-\frac{1}{2}}} I_n^H(t,x).$$

✠ When $H = \frac{1}{2}$ the following papers considered the estimator of $\theta$ by using 4-variation. However, when $H \neq \frac{1}{2}$ we shall need the $\frac{2}{2H-\frac{1}{2}}$-variation:

$$\sum_{j=1}^{n} \left| u^H(t_j,x) - u^H(t_{j-1},x) \right|^{\frac{4}{4H-1}}.$$

This kind of thinking creates computational difficulties.

[5] J. Pospisil and R. Tribe, Parameter estimation and exact variations for stochastic heat equations driven by space-time white noise, *Stoch. Anal. Appl.* **4** (2007), 830-856.

[6] I. Cialenco and Y. Huang, A note on parameter estimation for discretely sampled SPDEs, *Stochastics and Dynamics*, **20** (2020), No. 3, 2050016.

**Theorem (5)**

*Let the above assumptions hold and* $\hat{\theta}_n = \frac{n^{2H-\frac{3}{2}}}{K_H t^{2H-\frac{1}{2}}} I_n^H(t,x)$.

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- *When* $\frac{1}{2} < H < \frac{3}{4}$ *we have*

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \longrightarrow N(0, \lambda_H)$$

*in distribution, as* $n$ *tends to infinity.*

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- *When* $H = \frac{3}{4}$ *we have*

$$\sqrt{\frac{n}{\log n}}\left(\hat{\theta}_n - \theta\right) \longrightarrow N(0, \lambda)$$

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- *When* $\frac{3}{4} < H < 1$ *we have*

$$n^{2-2H}\left(\hat{\theta}_n - \theta\right) \longrightarrow \lambda'_H \mathscr{R}_H$$

  *in distribution, as* $n$ *tends to infinity, where* $\mathscr{R}_H$ *denotes the Rosenblatt random variable with variance* $1$ *and self-similarity parameter* $H$.

✠ Case 2: $h = h(n)$ satisfies the conditions

(C1) $h \downarrow 0$ and $t_n = nh \to +\infty$ as $n \to \infty$;

[11] Lv/Sun/Y., Quadratic covariations and parameter estimation of stochastic heat equation with additive time-space white noise, submitted 2023.

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**Theorem (6)**

*Fix $x \in \mathbb{R}$. Let the temporal process $u^x = \{u^H(t,x), t \geq 0\}$ is observed at some discrete time instants $\{t_j = jh, j = 0, 1, 2, \ldots, n\}$ with the conditions $(C1)$ and $(C2)$. The estimator*

$$\check{\theta}_n := K_H(1) n^{-\frac{3-4H+2\gamma}{2(1+\gamma)}} I_n(nh, x)$$

*is consistent and asymptotically unbiased.*

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*is consistent and asymptotically unbiased.*

- When $H = \frac{1}{2}$ we have

$$K_H(1) = \sqrt{\frac{\pi}{2}}.$$

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**Theorem (7)**

*Given* $\frac{1}{2} < H < \frac{3}{4}$. *Let the conditions in Theorem 6 hold and let there exist* $\alpha > 0$ *such that*

$$\lim_{n\to\infty} n^{\alpha}(nh^{1+\gamma} - 1) = \zeta \in (0, \infty).$$

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(1) *If $\alpha > \frac{1}{2}$, then*

$$n^{\frac{1}{2}}\left(\check{\theta}_n - \theta\right) \longrightarrow N\left(0, (\lambda''_H \theta)^2\right)$$

*in distribution, as $n$ tends to infinity.*

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*in distribution, as* $n$ *tends to infinity.*

(2) *If* $\alpha = \frac{1}{2}$, *then*

$$n^{\frac{1}{2}}\left(\check{\theta}_n - \theta\right) \longrightarrow N\left(\nu_H \theta, (\lambda_H'' \theta)^2\right)$$

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(3) *If $0 < \alpha < \frac{1}{2}$, then*

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*in $L^2$, as $n$ tends to infinity.*

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(2) *If $\alpha = \frac{1}{2}$, then*

$$n^{\frac{1}{2}}\left(\check{\theta}_n - \theta\right) \longrightarrow N\left(\nu_H\theta, (\lambda''_H \theta)^2\right)$$

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*in $L^2$, as $n$ tends to infinity.*

- When $H = \frac{1}{2}$ we have $\lambda''_H = \sqrt{2 + (2-\sqrt{2})^2 + \lambda}$ with

$$\lambda = \sum_{n=1}^{\infty}\left(\sqrt{n} - 2\sqrt{n+1} + \sqrt{n+2}\right)^2,$$

and $\nu_H = \frac{1}{2}(1+\gamma)^{-1}\zeta$.

**Theorem (8)**

*Given $H=\frac{3}{4}$. Let the conditions in Theorem 6 hold and let there exist $\alpha>0$ such that*

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$$n^{2-2H}\left(\hat{\theta}_n - \theta\right) \longrightarrow \theta\delta_H \mathscr{R}_H$$

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From the perspective of likelihood estimation and statistics, we can also establish the estimator of $\theta$ which is called the quasi-likelihood estimator.

✠ Consider the sample

$$\xi_j := u^H(t_j,x) - u^H(t_{j-1},x), \quad j = 1,2,\dots,n$$

and quasi-likelihood function

$$f(x_1,x_2,\dots,x_n) = \prod_{j=1}^{n} f_{\xi_j}(x_j),$$

where $f_{\xi_j}(x_j)$ is the density function of $\xi_j$. Then, by likelihood method we get the estimator of $\theta$ as follows:

$$\begin{aligned}\tilde{\theta}_n &= \frac{1}{n}\sum_{j=1}^{n}\frac{\{u^H(t_j,x)-u^H(t_{j-1},x)\}^2}{\sigma_j^2}\\ &= \frac{\sqrt{2\pi}}{H(2H-1)nh^{2H-\frac{1}{2}}}\sum_{j=1}^{n}\frac{\{u^H(t_j,x)-u^H(t_{j-1},x)\}^2}{\Delta_{j,j}-2\Delta_{j,j-1}+\Delta_{j-1,j-1}},\end{aligned}$$

where $\sigma_j^2$ is the variance of $u^H(t_j,x) - u^H(t_{j-1},x)$ with $\theta = 1$ and

$$\Delta_{i,j} = \int_0^i\int_0^j |u-v|^{2H-2}\frac{dvdu}{\sqrt{i+j-(u+v)}}.$$

- Results:

[9] S. Torres, C-A. Tudor and F-G. Viens, Quadratic variations for the fractional-colored stochastic heat equation, *Electron. J. Probab.* **19** (2014), no. 76, 1-51.

✠ Results:

(1) The estimator $\tilde{\theta}_n$ is unbiased;

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